

NONTANGENTIAL LIMITS AND FATOU-TYPE THEOREMS ON POST-CRITICALLY FINITE SELF-SIMILAR SETS

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ABSTRACT. In this paper we study the boundary limit properties of harmonic functions on $\mathbb{R}_+ \times K$, the solutions $u(t, x)$ to the Poisson equation

$$\frac{\partial^2 u}{\partial t^2} + \Delta u = 0,$$

where K is a p.c.f. set and Δ its Laplacian given by a regular harmonic structure. In particular, we prove the existence of nontangential limits of the corresponding Poisson integrals, and the analogous results of the classical Fatou theorems for bounded and nontangentially bounded harmonic functions.

1. INTRODUCTION

There has recently been a growing interest in the study of analysis on fractals, in particular post-critically finite (p.c.f.) self similar sets and their harmonic structure defined by Kigami [Kig93]. Analogous questions from classical analysis have been asked on the setting of p.c.f. fractals, from spectral theory of the Laplacian [ASST03, CSW09, ORS10], functional analysis [Str03, RS10, IR10] and differential equations [DSV99, Str05, Pel07].

In this paper we study the boundary limit properties of harmonic functions on the tube $\mathbb{R}_+ \times K$, the solutions $u(t, x)$ to the Poisson equation

$$\frac{\partial^2 u}{\partial t^2} + \Delta u = 0,$$

where K is a p.c.f. set and Δ its Laplacian given by a regular harmonic structure. In Section 3 we define the Poisson kernel and prove its elementary properties, as well as proving the existence of nontangential limits of Poisson integrals in the boundary $t \rightarrow 0$. Nontangential limits are defined in terms of proper "cones", depending on the Hausdorff dimension of K with respect to effective resistance metric.

In Section 4 we prove an analogous Fatou theorem for bounded Dirichlet harmonic functions on $\mathbb{R}_+ \times K$. We also extend these results to Dirichlet harmonic functions with uniformly bounded L^p norms, $1 \leq p < \infty$.

We finish the paper with the analogous version to the local Fatou theorem, in Section 5, for nontangentially bounded harmonic functions on $\mathbb{R}_+ \times K$. As we make use of estimates from below for the Neumann heat kernel on K , we prove this results only for nested fractals [Lin90].

Date: July 11, 2011.

2000 Mathematics Subject Classification. 28A80, 31B25.

Key words and phrases. Fractals, p.c.f. sets, Poisson integrals, boundary behavior of harmonic functions.

2. PRELIMINARIES

2.1. P.c.f. self-similar structures. Let $(K, S, \{F_i\}_{i \in S})$ be a self-similar structure. $W_m = S^m$ is the set of words of length m , and $W_* = \bigcup_{m \geq 0} W_m$, where $W_0 = \{\emptyset\}$ and \emptyset is called the empty word. For $w \in W_m$, we write $F_w = F_{w_1} \circ \cdots \circ F_{w_m}$ (F_\emptyset is set to be the identity mapping) and $K_w = F_w(K)$. The critical set of K is the set

$$\mathcal{C} = \bigcup_{\substack{i, j \in S \\ i \neq j}} K_i \cap K_j,$$

and the post-critical set is given by $V_0 = \{p \in K : \exists w \in W_*, F_w(p) \in \mathcal{C}\}$. We say that K is a post-critically finite (p.c.f.) self-similar structure if the set V_0 is finite. From now on, we will assume that K is a p.c.f. self-similar structure, and that $V_0 \neq \emptyset$.

V_0 will be called the boundary of K . We define $V_m = \bigcup_{w \in W_m} F_w(V_0)$, and $V_* = \bigcup_{m \geq 0} V_m$. Basic properties and examples of p.c.f. self-similar structures can be found in [Kig01, Section 1.3].

If $0 < \mu_i < 1$ and $\sum_{i \in S} \mu_i = 1$, let μ be the Bernoulli measure with weights $(\mu_i)_{i \in S}$. It satisfies $\mu(K_w) = \mu_w = \mu_{w_1} \cdots \mu_{w_m}$ for $w \in W_m$ and, for any integrable function f on K ,

$$\int_K f d\mu = \sum_{w \in W_m} \mu_w \int_K f \circ F_w d\mu.$$

We define the Banach spaces $L^p(K, \mu)$, $1 \leq p \leq \infty$, as usual.

2.2. Harmonic structure. Let (D, \mathbf{r}) be a regular harmonic structure on K , where $\mathbf{r} = (r_i)_{i \in S}$ with $0 < r_i < 1, i \in S$. We denote by $R(x, y)$ the effective resistance metric induced by (D, \mathbf{r}) . Under this metric, V_* is dense in K [Kig01, Section 3.3] and the functions F_w become contractions with Lipschitz constant $\text{Lip } F_w \leq r_w = r_{w_1} \cdots r_{w_m}$.

Let $\mathcal{E}(f, g)$ be the Dirichlet form associated to (D, \mathbf{r}) , defined on $\mathcal{F} \subset C(K)$. It satisfies, for $m \geq 1$,

$$\mathcal{E}(f, g) = \sum_{w \in W_m} \frac{1}{r_w} \mathcal{E}(f \circ F_w, g \circ F_w).$$

By a theorem of Kumagai [Kum93] (see [Kig01, Section 3.4]), for any self-similar measure μ on K , $(\mathcal{E}, \mathcal{F})$ is a local regular Dirichlet form on $L^2(K, \mu)$, and the corresponding non-negative self-adjoint operator H_N on $L^2(K, \mu)$ has compact resolvent. If we define $\mathcal{F}_0 = \{u \in \mathcal{F} : u|_{V_0} = 0\}$, then $(\mathcal{E}, \mathcal{F}_0)$ is a local Dirichlet form on $L^2(K, \mu)$, and the corresponding operator H_D has also compact resolvent. Moreover, H_D is invertible, and $(H_D)^{-1}$ is a compact operator on $L^2(K, \mu)$. The operators $-H_N$ and $-H_D$ are called the *Neumann* and *Dirichlet* Laplacians, respectively.

2.3. Maximal function. Let d be the unique real number that satisfies $\sum_{i \in S} (r_i)^d = 1$. d is called the *similarity dimension* of the harmonic structure (D, \mathbf{r}) , and it can be proved that d corresponds to the Hausdorff dimension of K with respect to the resistance metric R [Kig01, Section 4.2]. If μ is the self-similar measure on K with weights $\mu_i = (r_i)^d, i \in S$, it is easy to prove that $\mu(B_\varepsilon(x)) \sim \varepsilon^d$ for any $\varepsilon > 0$

sufficiently small. More precisely, there exist two constants $A_1, A_2 > 0$ such that, for any $x \in K$ and sufficiently small $\varepsilon > 0$,

$$(2.1) \quad A_1 \varepsilon^d \leq \mu(B_\varepsilon(x)) \leq A_2 \varepsilon^d.$$

See [S  e02] for details. For $f \in L^1(K, \mu)$, we define the operator

$$(2.2) \quad Mf(x) = \sup_{\varepsilon > 0} \frac{1}{\mu(B_\varepsilon(x))} \int_{B_\varepsilon(x)} |f| d\mu,$$

where $B_\varepsilon(x)$ is the ball of radius ε around x with respect to the effective resistance metric. The following theorem is standard [Ste70, Chapter III].

Theorem 2.1. *If f is an integrable function and Mf is given by (2.2), then $Mf(x)$ is finite a.e. Moreover,*

- (1) *There exists a constant $A > 0$ such that, for any $f \in L^1(K, \mu)$ and $\alpha > 0$,*

$$\mu(\{x \in K : Mf(x) > \alpha\}) \leq \frac{A}{\alpha} \|f\|_{L^1(K, \mu)};$$

- (2) *M extends to a bounded operator on $L^p(K, \mu)$ for $1 < p \leq \infty$.*

Remark 2.2. In the case where K is the interval $I = [0, 1]$, the functions F_1 and F_2 are the contractions $x \mapsto x/2$ and $x \mapsto x/2 + 1/2$, and the harmonic structure corresponds to $D = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}$ and $r_1 = r_2 = 1/2$, we have that the effective resistance metric R is equal to the standard metric and μ is the Lebesgue measure in I . In such case, (2.2) is the Hardy-Littlewood maximal operator and Theorem 2.1 is the classical maximal function theorem.

Remark 2.3. It is not hard to see that, as in the classical case, Theorem 2.1 is also true for finite Borel measures on K . Precisely, if ν is a finite Borel measure and we define the function $M\nu$ as

$$M\nu(x) = \sup_{\varepsilon > 0} \frac{1}{\mu(B_\varepsilon(x))} \int_{B_\varepsilon(x)} |d\nu|,$$

then

$$\mu(\{x \in K : M\nu(x) > \alpha\}) \leq \frac{A}{\alpha} \|\nu\|.$$

(Cf. [Ste70, Chapter III, 4.1].)

2.4. Laplacians. Let Δ be the Laplacian associated with (D, \mathbf{r}) and μ . We denote its domain by \mathcal{D} . We also consider the sets

$$\mathcal{D}_D = \{u \in \mathcal{D} : u|_{V_0} = 0\} \quad \text{and} \quad \mathcal{D}_N = \{u \in \mathcal{D} : du = 0 \text{ on } V_0\},$$

where $df(p)$ is the Neumann derivative of f at the boundary point p [Kig01]. One can see then that H_D is the Friedrich extension of $-\Delta$ on \mathcal{D}_D , while H_N is the Friedrich extension of $-\Delta$ on \mathcal{D}_N (see [Kig01, Section 3.7]).

For convenience, as in [Kig01], will denote by b either D or N ; so, for instance, H_b will denote the operator H_D or H_N , respectively.

2.5. Dirichlet and Neumann eigenfunctions. Consider the set $E_b(\lambda) = \{\phi \in \mathcal{D}_b : \Delta\phi = -\lambda\phi\}$. If $\dim E_D(\lambda) \neq 0$, λ is called a Dirichlet eigenvalue, and the collection of such λ is called the Dirichlet spectrum of Δ . If $\dim E_N(\lambda) \neq 0$, then λ is called a Neumann eigenvalue, and the collection of such λ is called the Neumann spectrum of Δ . Both the Dirichlet and Neumann spectra of Δ are subsets of $[0, \infty)$, and there are λ_n^b and $\phi_n^b \in E_b(\lambda_n^b)$ such that

$$0 \leq \lambda_1^b \leq \lambda_2^b \leq \dots$$

and $\{\phi_n^b : n \geq 1\}$ is a complete orthonormal system for $L^2(K, \mu)$. Observe that $\lambda_1^D > 0$. The following theorem is due to Kigami and Lapidus [KL93].

Theorem 2.4. *If $\rho_b(x) = \sum_{\lambda \leq x} \dim E_b(\lambda)$,*

$$(2.3) \quad 0 < \liminf_{x \rightarrow \infty} \frac{\rho_b(x)}{x^{d/(d+1)}} \leq \limsup_{x \rightarrow \infty} \frac{\rho_b(x)}{x^{d/(d+1)}} < \infty.$$

Equation (2.3) is the analogous to Weyl's formula which counts the eigenvalues of the Laplacian on a domain in \mathbb{R}^n . We will also make use of the following property:

* There exists a constant $C > 0$ such that

$$(2.4) \quad \|\phi\|_\infty \leq C \lambda^{\frac{d}{2(d+1)}} \|\phi\|_2, \quad \phi \in E_b(\lambda).$$

A proof of this can be found in [Kig01, Section 4.5].

3. THE HEAT AND POISSON KERNELS

3.1. Heat kernel. For $b = D$ or N , the Dirichlet (respectively Neumann) heat kernel is the function $H^b : \mathbb{R}_+ \times K \times K \rightarrow \mathbb{C}$ defined by

$$(3.1) \quad H^b(t, x, y) = \sum_{n=1}^{\infty} e^{-\lambda_n^b t} \phi_n^b(x) \phi_n^b(y).$$

Although the right-hand sum of (3.1) is defined only formally, it is not hard to prove that it converges uniformly on $[T, \infty) \times K \times K$ for any $T > 0$, which follows from the results in Section 2.5 (cf. Section 3.2). Moreover, H^b is nonnegative, continuous, defines a fundamental solution to the heat equation [Kig01, Section 5.1], and

$$(3.2) \quad \int_K H^b(t, x, z) H^b(s, z, y) d\mu(z) = H^b(t + s, x, y).$$

These properties imply that the operator $f \mapsto \mathcal{H}_t^b f$, where

$$\mathcal{H}_t^b f(x) = \int_K H^b(t, x, y) f(y) d\mu(y),$$

defined for $t > 0$ and integrable functions f on K , is a strongly continuous semigroup on $L^2(K, \mu)$ whose generator is given by $-H_b$. In fact, as H^b is continuous and K is compact, we have that \mathcal{H}_t^b is bounded from $L^p(K, \mu)$ to $C(K)$ for any $t > 0$ and $1 \leq p \leq \infty$ and, moreover,

$$\mathcal{H}_t^b(L^p(K, \mu)) \subset \mathcal{D}_b.$$

If $u(t, x)$ is defined on $\mathbb{R}_+ \times K$ as $u(t, x) = \mathcal{H}_t^b f(x)$, then

$$\frac{\partial u(t, x)}{\partial t} = \Delta u(t, x),$$

so H^b is the fundamental solution of the heat equation on K . For the details of these facts, see [Kig01, Chapter 5]. The following result is also proved in [Kig01, Proposition 5.2.6].

Proposition 3.1. (1) For $f \in C(K)$, $\|\mathcal{H}_t^N f - f\|_{L^\infty(K, d\mu)} \rightarrow 0$ as $t \rightarrow 0$.
 (2) Let $f \in C(K)$, with $f|_{V_0} \equiv 0$. Then $\|\mathcal{H}_t^D f - f\|_{L^\infty(K, d\mu)} \rightarrow 0$ as $t \rightarrow 0$.

Thus we observe that the heat kernel acts as an *approximation to the identity* for continuous functions.

3.2. Poisson kernel. We define the *Dirichlet* (respectively *Neumann*) *Poisson kernel* $P^b : \mathbb{R}_+ \times K \times K \rightarrow \mathbb{C}$ by

$$(3.3) \quad P^b(t, x, y) = \sum_{n=1}^{\infty} e^{-\sqrt{\lambda_n^b} t} \phi_n^b(x) \phi_n^b(y).$$

As above, we first take the series in (3.3) formally. However, the series converges uniformly on $[T, \infty) \times K \times K$ for any $T > 0$, which follows from the following two observations.

(1) There exist $c_1, c_2 > 0$ such that

$$(3.4) \quad c_1 n^{(d+1)/d} \leq \lambda_n^b \leq c_2 n^{(d+1)/d},$$

(2) For $\alpha, \beta, \gamma, T > 0$, $\sum_{n \geq 1} n^\alpha e^{-\gamma n^\beta t}$ converges uniformly for $t \in [T, \infty)$.

The first follows from Theorem 2.4, while the second is straightforward from the well-known properties of the exponential function. Now, from these and equation (2.4) it follows that

$$|e^{-\sqrt{\lambda_n^b} t} \phi_n^b(x) \phi_n^b(y)| \lesssim e^{-\sqrt{c_1} t n^{(d+1)/(2d)}} n$$

uniformly in $K \times K$, and thus the series in (3.3) converges uniformly on $[T, \infty) \times K \times K$ for any $T > 0$.

The following identity, based on the *principle of subordination* [Ste70, Section III.2], will be useful to study the properties of $P^b(t, x, y)$.

Proposition 3.2. For $(t, x, y) \in \mathbb{R}_+ \times K \times K$,

$$P^b(t, x, y) = \frac{t}{2\sqrt{\pi}} \int_0^\infty e^{-t^2/4s} H^b(s, x, y) \frac{ds}{s^{3/2}},$$

where H^b is the heat kernel on K .

Proof. By the estimates (2.4) and (3.4), we see that

$$H^b(t, x, y) \leq \sum_{n=1}^{\infty} e^{-\lambda_n^b t} (\lambda_n^b)^{d/(d+1)} \leq \sum_{n=1}^{\infty} n e^{-c_1 n^{(d+1)/d} t},$$

uniformly on $K \times K$. Let $\alpha = \frac{d}{(d+1)}$. Since, $e^{-c_1 n^{1/\alpha} t} \lesssim \frac{1}{(n^{1/\alpha} t)^{2\alpha+1}}$, we see that the series can be estimated by

$$\sum_{n=1}^{\infty} n e^{-c_1 n^{1/\alpha} t} \lesssim \frac{1}{t^{2\alpha+1}}.$$

Now the integral $\int_0^\infty e^{-t^2/4s} \frac{1}{s^{2\alpha+1}} \frac{ds}{s^{3/2}}$ converges for $t > 0$, so by the dominated convergence theorem

$$\begin{aligned} \int_0^\infty e^{-t^2/4s} H^b(s, x, y) \frac{ds}{s^{3/2}} &= \sum_{n=1}^\infty \phi_n^b(x) \phi_n^b(y) \int_0^\infty e^{-t^2/4s} e^{-\lambda_n^b s} \frac{ds}{s^{3/2}} \\ &= \sum_{n=1}^\infty \phi_n^b(x) \phi_n^b(y) \frac{2\sqrt{\pi}}{t} e^{-\sqrt{\lambda_n^b} t} = \frac{2\sqrt{\pi}}{t} P^b(t, x, y), \end{aligned}$$

where we have used the identity

$$\int_0^\infty e^{-t^2/4s} e^{-\beta^2 s} \frac{ds}{s^{3/2}} = \frac{2\sqrt{\pi}}{t} e^{-\beta t}$$

for $\beta > 0$ [Ste70, Section III.2]. \square

Proposition 3.2 allows us to conclude the following properties of $P^b(t, x, y)$, analogously to those of $H^b(t, x, y)$.

Corollary 3.3. *The Poisson kernel satisfies the following properties.*

- (1) P^b is nonnegative and continuous;
- (2) For $(t, x) \in \mathbb{R}_+ \times K$, $P^b(t, x, \cdot) \in \mathcal{D}_b$;
- (3) For $(x, y) \in K \times K$, $P^b(\cdot, x, y) \in C^2(\mathbb{R}_+)$;
- (4) For $(t, x, y) \in \mathbb{R}_+ \times K \times K$,

$$(3.5) \quad \frac{\partial^2 P^b(t, x, y)}{\partial t^2} + (\Delta P^b(t, x, \cdot))(y) = 0;$$

and

- (5) For $t, s \in \mathbb{R}_+$, $x, y \in K$,

$$(3.6) \quad \int_K P^b(t, x, z) P^b(s, z, y) d\mu(z) = P^b(t + s, x, y).$$

Proof. (1) follows from the nonnegativity of $H^b(t, x, y)$ and the fact that the integral in Proposition 3.2 converges absolutely.

To prove (2), observe that $\sum_{n \geq 1} a_n \phi_n^b \in \text{dom}(H_b)$ if and only if

$$\sum_{n \geq 1} |\lambda_n^b a_n|^2 < \infty,$$

which clearly holds for $a_n = e^{-\sqrt{\lambda_n^b} t} \phi_n^b(x)$ for any given $t > 0$ and $x \in K$, because of (2.4). Then $P^b(t, x, \cdot) \in \text{dom}(H_b)$ and

$$H_b P^b(t, x, \cdot) = \sum_{n=1}^\infty \lambda_n^b e^{-\sqrt{\lambda_n^b} t} \phi_n^b(x) \phi_n^b.$$

As above, one can verify that this series converges uniformly on $[T, \infty) \times K \times K$, and thus $H_b P^b(t, x, \cdot) \in C(K)$. Therefore $P^b(t, x, \cdot) \in \mathcal{D}_b$.

Now fix $x, y \in K$ and set $f_n(t) = e^{-\sqrt{\lambda_n^b} t} \phi_n^b(x) \phi_n^b(y)$. Since

$$\sum_{n=1}^\infty f_n'(t) = - \sum_{n=1}^\infty \sqrt{\lambda_n^b} e^{-\sqrt{\lambda_n^b} t} \phi_n^b(x) \phi_n^b(y)$$

converges uniformly on $[T, \infty)$ for any $T > 0$, $t \mapsto P^b(t, x, y)$ is continuously differentiable and

$$\frac{\partial P^b(t, x, y)}{\partial t} = \sum_{n=1}^{\infty} f'_n(t).$$

Now $f''_n(t) = \lambda_n^b e^{-\sqrt{\lambda_n^b} t} \phi_n^b(x) \phi_n^b(y)$, so $\sum_{n=1}^{\infty} f''_n(t)$ also converges absolutely on $[T, \infty)$ for any $T > 0$ and thus $t \mapsto P^b(t, x, y)$ is in $C^2(\mathbb{R}_+)$, which proves (3).

(4) follows from the fact that

$$\begin{aligned} \frac{\partial^2 P^b(t, x, y)}{\partial t^2} &= \sum_{n=1}^{\infty} f''_n(t) = \sum_{n=1}^{\infty} \lambda_n^b e^{-\sqrt{\lambda_n^b} t} \phi_n^b(x) \phi_n^b(y) \\ &= (H_b P^b(t, x, \cdot))(y) = -(\Delta P^b(t, x, \cdot))(y). \end{aligned}$$

For (5), it is sufficient to note that (3.6) follows from (3.2), Proposition 3.2 and Fubini's theorem. \square

3.3. Poisson semigroup. The results from the previous section lead us, analogously to the heat kernel, to define the operators $f \mapsto \mathcal{P}_t^b f$ ($b = D$ or N) for each $t > 0$ as

$$(3.7) \quad \mathcal{P}_t^b f(x) = \int_K P^b(t, x, y) f(y) d\mu(y),$$

defined for integrable functions f on K . The continuity of P^b and the compactness of K imply that \mathcal{P}_t^b is bounded from $L^p(K, \mu)$ to $C(K)$ for any $t > 0$ and $1 \leq p \leq \infty$. We also see that $\mathcal{P}_t^b \circ \mathcal{P}_s^b = \mathcal{P}_{t+s}^b$, which follows from (3.6), so we have that $\{\mathcal{P}_t^b\}_{t>0}$ is a semigroup.

We in fact have the following theorem.

Theorem 3.4. *Let $f \in L^p(K, \mu)$ and define, for $(t, x) \in K$,*

$$u(t, x) = \mathcal{P}_t^b f(x).$$

- (1) *For each $x \in K$, $u(\cdot, x) \in C^\infty(\mathbb{R}_+)$;*
- (2) *For each $t > 0$, $u(t, \cdot) \in \mathcal{D}_b$; and*
- (3) *For each $(t, x) \in \mathbb{R}_+ \times K$,*

$$\frac{\partial^2 u(t, x)}{\partial t^2} + \Delta u(t, x) = 0.$$

Proof. Since, for $f \in L^p$,

$$\int_K |\phi_n^b(y) f(y)| d\mu(y) \leq C \lambda_n^{\frac{d}{2(d+1)}} \|f\|_{L^p}$$

by (2.4), we have that, for every positive integer m ,

$$\left\| \sum_{n=1}^m e^{-\sqrt{\lambda_n^b} t} \phi_n^b(x) \phi_n^b f \right\|_{L^1} \leq C \|f\|_{L^p} \sum_{n=1}^{\infty} e^{-\sqrt{\lambda_n^b} t} (\lambda_n^b)^{\frac{d}{d+1}},$$

so by (3.4) the sum is uniformly bounded in m by $C' \|f\|_{L^p}$. The dominated convergence theorem implies then that $u(t, x) = \sum_{n=1}^{\infty} a_n e^{-\sqrt{\lambda_n^b} t} \phi_n^b(x)$, where

$$a_n = \int_K \phi_n^b(y) f(y) d\mu(y).$$

Now fix $x \in K$ and set $\psi_n(t) = a_n e^{-\sqrt{\lambda_n^b} t} \phi_n^b(x)$. For each $k \in \mathbb{N}$,

$$\psi^{(k)}(t) = a_n (\lambda_n^b)^{k/2} e^{-\sqrt{\lambda_n^b} t} \phi_n^b(x),$$

so the series $\sum \psi_n^{(k)}(t)$ converges uniformly for $t \in [T, \infty)$, for any $T > 0$, by (3.4). It follows that $t \mapsto u(t, x)$ is in $C^k(\mathbb{R}_+)$ for any k , which proves (1).

We now fix $t > 0$. For (2), we first need to verify that

$$\sum_{n=1}^{\infty} |a_n e^{-\sqrt{\lambda_n^b} t}|^2 < \infty.$$

But this, again, follows from (3.4) and the fact that $|a_n| \leq C(\lambda_n^b)^{\frac{d}{2(d+1)}} \|f\|_{L^p}$. This shows $x \mapsto u(t, x)$ is in $\text{dom}(H_b)$ and

$$H_b u(t, \cdot) = \sum_{n=1}^{\infty} a_n \lambda_n^b e^{-\sqrt{\lambda_n^b} t} \phi_n^b.$$

As above, this series converges uniformly, so we have that $H_b u(t, \cdot) \in C(K)$, and we conclude $u(t, \cdot) \in \mathcal{D}_b$.

Part (3) follows from the sequence of identities

$$\frac{\partial^2 u(t, x)}{\partial t^2} = \sum_{n=1}^{\infty} \psi_n''(t) = \sum_{n=1}^{\infty} a_n \lambda_n^b e^{-\sqrt{\lambda_n^b} t} \phi_n^b(x) = H_b u(t, x) = -\Delta u(t, x).$$

□

We will call the function $u(t, x) = \mathcal{P}_t^b f(x)$ the *Dirichlet Poisson integral* (respectively *Neumann Poisson integral*) of f .

As in the case of the heat semigroup, it is not hard to see that $f \mapsto \mathcal{P}_t^b f$ is a strongly continuous semigroup on $L^2(K, \mu)$, which implies that $u(t, x) \rightarrow f(x)$ as $t \rightarrow 0$ in L^2 . It also acts as an approximation to the identity on continuous functions.

Proposition 3.5. (1) *Let $f \in C(K)$ and $u_N(t, x)$ its Neumann Poisson integral. Then $\|u_N(t, \cdot) - f\|_{L^\infty(K, d\mu)} \rightarrow 0$ as $t \rightarrow 0$.*

(2) *Let $f \in C(K)$ with $f|_{V_0} \equiv 0$ and $u_D(t, x)$ its Dirichlet Poisson integral. Then $\|u_D(t, \cdot) - f\|_{L^\infty(K, d\mu)} \rightarrow 0$ as $t \rightarrow 0$.*

Proof. This proposition follows from Propositions 3.1 and 3.2. Indeed, as

$$\frac{t}{2\sqrt{\pi}} \int_0^\infty e^{-t^2/4s} \frac{ds}{s^{3/2}} = \frac{t}{2\sqrt{\pi}} \int_0^\infty e^{-\frac{t^2}{4}s} s^{1/2} \frac{ds}{s} = 1,$$

using Fubini's theorem we obtain

$$u_b(t, x) - f(x) = \frac{t}{2\sqrt{\pi}} \int_0^\infty e^{-t^2/4s} (\mathcal{H}_s^b f(x) - f(x)) \frac{ds}{s^{3/2}},$$

where \mathcal{H}_s^b is the heat semigroup. By Proposition 3.1, for any $f \in C(K)$ and any $\varepsilon > 0$ there is $\delta > 0$ such that, if $0 < s < \delta$,

$$\|\mathcal{H}_s^b f - f\|_{L^\infty(K, d\mu)} < \varepsilon,$$

if $b = N$, and for $f \in C(K)$ with $f|_{V_0} \equiv 0$ if $b = D$. Thus

$$\begin{aligned} \|u_b(t, \cdot) - f\|_{L^\infty(K, d\mu)} &\leq \frac{t}{2\sqrt{\pi}} \int_0^\infty e^{-t^2/4s} \|\mathcal{H}_s^b f - f\|_{L^\infty(K, d\mu)} \frac{ds}{s^{3/2}} \\ &\leq \frac{t}{2\sqrt{\pi}} \int_0^\delta e^{-t^2/4s} \frac{ds}{s^{3/2}} + \frac{t}{2\sqrt{\pi}} \int_\delta^\infty e^{-t^2/4s} M \frac{ds}{s^{3/2}}, \end{aligned}$$

where $M > 0$ is such that $\|\mathcal{H}_s^b f - f\|_{L^\infty(K, d\mu)} \leq M$ uniformly in s (f is continuous).

Now

$$\frac{t}{2\sqrt{\pi}} \int_0^\delta e^{-t^2/4s} \frac{ds}{s^{3/2}} \leq \frac{t}{2\sqrt{\pi}} \int_0^\infty e^{-t^2/4s} \frac{ds}{s^{3/2}} = 1,$$

and

$$\frac{t}{2\sqrt{\pi}} \int_\delta^\infty e^{-t^2/4s} \frac{ds}{s^{3/2}} \leq \frac{t}{2\delta^{1/4}\sqrt{\pi}} \int_0^{1/\delta} e^{-t^2 s/4} s^{1/4} \frac{ds}{s} \leq \frac{\sqrt{t}}{\sqrt{2}\delta^{1/4}\sqrt{\pi}} \Gamma(1/4),$$

so

$$\|u_b(t, \cdot) - f\|_{L^\infty(K, d\mu)} \leq \varepsilon + C\sqrt{t}$$

for some constant $C > 0$. Therefore, as $\varepsilon > 0$ is arbitrary, we obtain both cases of the proposition. \square

We now state and prove the following theorem, which describes the boundary behavior of $u(t, x)$ for $f \in L^p$.

Theorem 3.6. *Let $f \in L^p(K, \mu)$, $1 \leq p \leq \infty$, and $u(t, x)$ either its Dirichlet or Neumann Poisson integral.*

(1) *There exists a constant $A > 0$ such that, for every $t > 0$,*

$$|u(t, x)| \leq AMf(x),$$

where Mf is the maximal function defined in Section 2.3;

(2) *$u(t, \cdot) \rightarrow f$ in $L^p(K, \mu)$, if $1 \leq p < \infty$;*

(3) *$\lim_{t \rightarrow 0} u(t, x) = f(x)$ for a.e. $x \in K$.*

For the proof of this theorem we use the following Lemma.

Lemma 3.7. *There exists a constant $C > 0$ such that, for any $x, y \in K$ and $t > 0$,*

$$P^b(t, x, y) \leq C \min \left\{ t^{-\frac{2d}{d+1}}, \frac{t}{R(x, y)^{\frac{3d+1}{2}}} \right\}.$$

Proof. This lemma follows from the estimate for the heat kernel

$$H^b(t, x, y) \leq At^{-\frac{d}{d+1}} \exp \left(-c' \left(\frac{R(x, y)^{d+1}}{t} \right)^{1/d} \right)$$

for some $A, c' > 0$ [Bar98, Theorem 8.15]. By Proposition 3.2,

$$P^b(t, x, y) = \frac{t}{2\sqrt{\pi}} \int_0^\infty e^{-t^2/4s} H^b(s, x, y) \frac{ds}{s^{3/2}},$$

so we have

$$\begin{aligned} P^b(t, x, y) &\lesssim t \int_0^\infty e^{-t^2/4s} s^{-\frac{d}{d+1}} \frac{ds}{s^{3/2}} = t \int_0^\infty e^{-t^2/4s} s^{-\frac{3d+1}{2(d+1)}} \frac{ds}{s} \\ &\approx t \cdot t^{-\frac{3d+1}{d+1}} = t^{-\frac{2d}{d+1}}, \end{aligned}$$

and also

$$P^b(t, x, y) \lesssim t \int_0^\infty \exp\left(-c' \left(\frac{R(x, y)^{d+1}}{s}\right)^{1/d}\right) s^{-\frac{3d+1}{2(d+1)}} \frac{ds}{s} \lesssim t \cdot R(x, y)^{-\frac{3d+1}{2}}.$$

□

Proof of Theorem 3.6. (1) This part follows by an argument analogous to the well-known Euclidean case: since the Poisson integral $u(t, x)$ of f is given by

$$u(t, x) = \int_K P^b(t, x, y) f(y) d\mu(y),$$

we write

$$|u(t, x)| \leq \int_K P^b(t, x, y) |f(y)| d\mu(y) = \sum_{n=0}^\infty \int_{A_n(x)} P^b(t, x, y) |f(y)| d\mu(y),$$

where $A_0(x) = \{y \in K : R(x, y) \leq t^{\frac{2}{d+1}}\}$, and

$$A_n(x) = \{y \in K : 2^{n-1} t^{\frac{2}{d+1}} < R(x, y) \leq 2^n t^{\frac{2}{d+1}}\}, \quad n \geq 1.$$

Now, from Lemma 3.7 and the estimate (2.1),

$$\begin{aligned} \int_{A_0(x)} P^b(t, x, y) |f(y)| d\mu(y) &\lesssim t^{-\frac{2d}{d+1}} \cdot \frac{t^{\frac{2}{d+1} \cdot d}}{\mu(B_{t^{2/(d+1)}}(x))} \int_{B_{t^{2/(d+1)}}(x)} |f(y)| d\mu(y) \\ &\leq Mf(x). \end{aligned}$$

Similarly, for each $n \geq 1$,

$$\begin{aligned} \int_{A_n(x)} P^b(t, x, y) |f(y)| d\mu(y) &\lesssim t \int_{A_n(x)} \frac{1}{R(x, y)^{\frac{3d+1}{2}}} |f(y)| d\mu(y) \\ &\leq t \cdot \frac{1}{(2^{n-1} t^{\frac{2}{d+1}})^{\frac{3d+1}{2}}} \int_{B_{2^n t^{2/(d+1)}}(x)} |f(y)| d\mu(y) \\ &\lesssim \frac{t^{-\frac{2d}{d+1}} 2^{-n \frac{3d+1}{2}} (2^n t^{\frac{2}{d+1}})^d}{\mu(B_{2^n t^{2/(d+1)}}(x))} \int_{B_{2^n t^{2/(d+1)}}(x)} |f(y)| d\mu(y) \\ &\leq 2^{-\frac{d+1}{2}n} Mf(x). \end{aligned}$$

Therefore

$$|u(t, x)| \lesssim \sum_{n=0}^\infty 2^{-\frac{d+1}{2}n} Mf(x) \lesssim Mf(x).$$

- (2) This part follows, as in the classical case, from the fact that the family $\{P^b(t, x, y)\}_{t>0}$ forms an approximation to the identity for continuous functions, and the fact that the maximal function is weakly bounded in L^1 and bounded in L^p , $p > 1$.

- (3) This is standard [SW71].

□

Remark 3.8. We observe that we can write the estimates of Lemma 3.7 as

$$P^b(t, x, y) \leq \frac{C't}{(t^2 + R(x, y)^{d+1})^{\frac{3d+1}{2(d+1)}}},$$

for some $C' > 0$ and either $b = N$ or D , and thus we have an analogous inequality to the classical Poisson kernel.

Remark 3.9. If ν is a finite Borel measure on K , we can define its Dirichlet and Neumann Poisson integrals $u(t, x) = \mathcal{P}_t^b \nu(x)$ as

$$\mathcal{P}_t^b \nu(x) = \int_K P^b(t, x, y) d\nu(y).$$

The same arguments as in Theorem 3.6(1) imply the estimate $|u(t, x)| \lesssim M\nu(x)$, and hence $u(t, \cdot) \rightarrow \nu$ as $t \rightarrow 0$. In particular, $\|u(t, \cdot)\|_{L^1}$ is uniformly bounded.

3.4. Nontangential limits. In this section we discuss nontangential limits of Poisson integrals. We first develop the concept of a *cone* over a point $x \in K$. From the final remark in the previous section, for $\alpha > 0$ we consider the set

$$\Gamma_\alpha(x) = \{(t, y) \in \mathbb{R}_+ \times K : R(x, y)^{d+1} < \alpha t^2\}.$$

The set $\Gamma_\alpha(x)$ is not properly a cone; however, in the case $d > 1$, it contains the intersection of the cone

$$\{(t, y) \in \mathbb{R}_+ \times K : R(x, y) < \sqrt{\alpha t}\}$$

with set $\{(t, y) \in \mathbb{R}_+ \times K : R(x, y) < 1\}$, as $R(x, y)^2 > R(x, y)^{d+1}$ for such points.

We now state the following result.

Theorem 3.10. *Let $f \in L^p(K, d\mu)$, $1 \leq p \leq \infty$, and $u(t, x)$ either its Dirichlet or its Neumann Poisson integral. Let $\alpha > 0$. Then*

- (1) *There exists $A_\alpha > 0$ such that, for $x \in K$,*

$$\sup_{(t, y) \in \Gamma_\alpha(x)} |u(t, y)| \leq A_\alpha Mf(x),$$

where Mf is the maximal function of f ;

- (2) *For almost every $x \in K$,*

$$\lim_{\substack{(t, y) \rightarrow (0, x) \\ (t, y) \in \Gamma_\alpha(x)}} u(t, y) = f(x).$$

Proof. (1) The proof of (1) follows as the one in Theorem 3.6, once we prove that, for $(t, y) \in \Gamma_\alpha(x)$, $P^b(t, y, z)$ satisfies an estimate as in Lemma 3.7 for (t, x, z) , i. e.

$$(3.8) \quad P^b(t, y, z) \leq C_\alpha \min \left\{ t^{-\frac{2d}{d+1}}, \frac{t}{R(x, z)^{\frac{3d+1}{2}}} \right\},$$

for some constant $C_\alpha > 0$. Indeed, if $R(x, y) \leq \frac{1}{2}R(x, z)$,

$$R(y, z) \geq R(x, z) - R(x, y) \geq \frac{1}{2}R(x, z),$$

and we have, by Lemma 3.7,

$$P^b(t, y, z) \leq \frac{Ct}{R(y, z)^{\frac{3d+1}{2}}} \leq \frac{C't}{R(x, z)^{\frac{3d+1}{2}}}.$$

If $R(x, y) > \frac{1}{2}R(x, z)$, since $(t, y) \in \Gamma_\alpha(x)$,

$$\alpha t^2 > R(x, y)^{d+1} > \frac{1}{2^{d+1}} R(x, z)^{d+1},$$

and therefore, by Lemma 3.7

$$\begin{aligned} P^b(t, y, z) &\leq Ct^{-\frac{2d}{d+1}} = \frac{Ct}{t^{\frac{3d+1}{d+1}}} \\ &< \frac{Ct}{\left(\frac{1}{\alpha 2^{d+1}} (R(x, z)^{d+1})\right)^{\frac{3d+1}{2(d+1)}}} = \frac{C_\alpha t}{R(x, z)^{\frac{3d+1}{2}}}. \end{aligned}$$

The proof now follows as in Theorem 3.6, by decomposing the Poisson integral

$$u(t, y) = \int_K P^b(t, y, z) f(z) d\mu(z)$$

in annuli $A_n(x)$, with center x , of radius $\sim 2^n t^{\frac{2}{d+1}}$ for each n . □

For the second part of the theorem we need the following lemma.

Lemma 3.11. *For any $x \in K \setminus V_0$,*

$$\lim_{t \rightarrow 0} \int_K P^D(t, x, y) d\mu(y) = 1.$$

Proof. Let U be a neighborhood of V_0 and $\delta > 0$ such that $B_\delta(x) \cap U = \emptyset$. Let f be a continuous function on K such that $f \equiv 1$ on $K \setminus U$ and $f|_{V_0} \equiv 0$. Hence

$$\int_K P^D(t, x, y) d\mu(y) = \int_K P^D(t, x, y) f(y) d\mu(y) + \int_U P^D(t, x, y) (1 - f(y)) d\mu(y).$$

By Proposition 3.5,

$$\int_K P^D(t, x, y) f(y) d\mu(y) \rightarrow f(x) = 1$$

as $t \rightarrow 0$ and, since $R(x, y) > \delta$ for $y \in U$,

$$\int_U P^D(t, x, y) |1 - f(y)| d\mu(y) \leq At \int_U \frac{d\mu(y)}{R(x, y)^{\frac{3d+1}{2}}} < A_\delta \mu(U) t.$$

Therefore

$$\int_K P^D(t, x, y) d\mu(y) \rightarrow 1$$

as $t \rightarrow 0$. □

Observe that in the Neumann case one actually has

$$\int_K P^N(t, x, y) d\mu(y) = 1$$

for every $t > 0$.

Proof of Theorem 3.10. (2) Let $x \in K \setminus V_0$ be in the Lebesgue set of f , and let $\varepsilon > 0$. Thus, there exists $\delta > 0$ such that

$$(3.9) \quad \frac{1}{\mu(B_r(x))} \int_{B_r(x)} |f(y) - f(x)| d\mu(y) < \varepsilon$$

for every $r < \delta$. We thus define the function g on K by

$$g(y) = \begin{cases} |f(y) - f(x)| & R(x, y) < \delta \\ 0 & R(x, y) \geq \delta. \end{cases}$$

Hence (3.9) implies $Mg(x) < \varepsilon$. We now have

$$|u(t, y) - f(x)| \leq \int_K P^b(t, y, z) |f(z) - f(x)| d\mu(z) + \left| \int_K P^b(t, y, z) d\mu(z) - 1 \right| \cdot |f(x)|.$$

By Lemma 3.11, the second term in the right hand side is either zero in the Neumann case $b = N$, or goes to 0 as $t \rightarrow 0$ in the Dirichlet case $b = D$.

We split the first term as the sum

$$\int_{B_\delta(x)} P^b(t, y, z) |f(z) - f(x)| d\mu(z) + \int_{K \setminus B_\delta(x)} P^b(t, y, z) |f(z) - f(x)| d\mu(z).$$

Now, by part (1) of the theorem, if $(t, y) \in \Gamma_\alpha(x)$,

$$\begin{aligned} \int_{B_\delta(x)} P^b(t, y, z) |f(z) - f(x)| d\mu(z) &= \int_K P^b(t, y, z) g(z) d\mu(z) \\ &\leq A_\alpha Mg(x) < A_\alpha \varepsilon. \end{aligned}$$

Also, by (3.8), if $(t, y) \in \Gamma_\alpha(x)$,

$$\begin{aligned} \int_{K \setminus B_\delta(x)} P^b(t, y, z) |f(z) - f(x)| d\mu(z) &\leq C_\alpha t \int_{K \setminus B_\delta(x)} \frac{|f(z) - f(x)|}{R(x, z)^{\frac{3d+1}{2}}} d\mu(z) \\ &\leq C_{\alpha, \delta} t \int_K |f(z) - f(x)| d\mu(z) \\ &\leq C_{\alpha, \delta} (\|f\|_{L^p} + |f(x)|) t, \end{aligned}$$

and thus goes to 0 as $t \rightarrow 0$. Hence we have

$$\limsup_{\substack{(t, y) \rightarrow (0, x) \\ (t, y) \in \Gamma_\alpha(x)}} |u(t, y) - f(x)| \leq A_\alpha \varepsilon$$

and, as $\varepsilon > 0$ is arbitrary,

$$\lim_{\substack{(t, y) \rightarrow (0, x) \\ (t, y) \in \Gamma_\alpha(x)}} u(t, y) = f(x).$$

Since V_0 is finite and the Lebesgue set of f contains almost every point of K (if $f \in L^p(K, d\mu)$, then f is integrable, as we have noted above), therefore we obtain the theorem. \square

4. FATOU-TYPE THEOREMS

4.1. A maximum principle. We say that a continuous function u on $\mathbb{R}_+ \times K$ is a *harmonic function* if

- (1) $u(\cdot, x) \in C^2(\mathbb{R}_+)$ for each $x \in K$;
- (2) $u(t, \cdot) \in \mathcal{D}$ for each $t > 0$; and
- (3) For each $(t, x) \in \mathbb{R}_+ \times K \setminus V_0$,

$$(4.1) \quad \frac{\partial^2 u(t, x)}{\partial t^2} + \Delta u(t, x) = 0.$$

For example, if $u(t, x)$ is the Poisson integral of $f \in L^p(K, d\mu)$, then u is a harmonic function, by Theorem 3.4.

We state and prove the following result, analogous to the *parabolic* maximum principle for solutions of the heat equation [Kig01, Section 5.2].

Theorem 4.1. *Let u be a harmonic function on $\mathbb{R}_+ \times K$. Then u cannot take a maximum in $\mathbb{R}_+ \times (K \setminus V_0)$.*

As in the case of the parabolic maximum principle, the proof of Theorem 4.1 makes use of the following lemma, whose proof can be found in [Kig01, Lemma 5.2.4].

Lemma 4.2. *Let $u \in \mathcal{D}$. If $u(x) = \max\{u(y) : y \in K\}$ for some $x \in K \setminus V_0$, then $\Delta u(x) \leq 0$.*

Proof of Theorem 4.1. For $n \in \mathbb{Z}_+$, set $u_n = u + t^2/n$. Suppose u_n takes its maximum at an interior point $(t_0, x_0) \in \mathbb{R}_+ \times (K \setminus V_0)$. Then

$$\frac{\partial^2 u_n}{\partial t^2}(t_0, x_0) \leq 0.$$

But

$$\frac{\partial^2 u_n}{\partial t^2} = \frac{\partial^2 u}{\partial t^2} + \frac{2}{n} = -\Delta u + \frac{2}{n},$$

so

$$\Delta u(t_0, x_0) = -\frac{\partial^2 u_n}{\partial t^2}(t_0, x_0) + \frac{2}{n} > 0,$$

contradicting Lemma 4.2 because the function

$$x \mapsto u(t_0, x) = u_n(t_0, x) - \frac{t_0^2}{n}$$

takes its maximum at $x_0 \in K \setminus V_0$. □

From Theorem 4.1, if u is a bounded continuous function on $[0, \infty) \times K$ which is harmonic on $\mathbb{R}_+ \times K$, then, if u takes its maximum in (t_0, x_0) , then $t_0 = 0$ or $x_0 \in V_0$.

Moreover, by taking $-u$, we can similarly conclude that its minimum, if taken by u at (t_0, x_0) , must satisfy $t_0 = 0$ or $x_0 \in V_0$. We have the following corollary.

Corollary 4.3. *Let u be continuous in $[0, \infty) \times K$, harmonic on $\mathbb{R}_+ \times K$, and $0 \leq a < b$. If $u(a, x) \geq 0$ and $u(b, x) \geq 0$ for every $x \in K$, and $u(t, p) \geq 0$ for every $p \in V_0$ and $a \leq t \leq b$, then $u(t, x) \geq 0$ for every $(t, x) \in [a, b] \times K$.*

We can clearly extend this corollary to any open set in $\mathbb{R}_+ \times K$.

Corollary 4.4. *Let u be continuous in $[0, \infty) \times K$, harmonic on $\mathbb{R}_+ \times K$ and $\Omega \subset \mathbb{R}_+ \times K$ an open set. If $u(t, x) \geq 0$ for $(t, x) \in \partial\Omega$, then $u(t, x) \geq 0$ for every $(t, x) \in \Omega$.*

4.2. Harmonic functions and boundary limits. We have observed that, if $u(t, x)$ is either the Dirichlet or Neumann Poisson integral of $f \in L^p(K, d\mu)$, then u is a harmonic function on $\mathbb{R}_+ \times K$. Moreover, Theorem 3.6 implies that

$$\sup_{t>0} \|u(t, \cdot)\|_{L^p(K, d\mu)} < \infty$$

if $1 \leq p \leq \infty$. We now prove the converse for uniformly bounded Dirichlet harmonic functions, a result analogous to the classical Fatou's theorem [ABR01].

One can easily observe that if $u(t, x)$ is the Dirichlet Poisson integral of a function on K , then $u(t, p) = 0$ for every $p \in V_0$ and $t > 0$, as the Dirichlet Poisson kernel satisfies $P^D(t, p, y) = 0$ for $p \in V_0$.

We say that a continuous function u on $\mathbb{R}_+ \times K$ is a *Dirichlet harmonic function* if u is harmonic and $u(t, p) = 0$ for every $t > 0$ and $p \in V_0$.

Theorem 4.5. *Let u be a Dirichlet harmonic function on $\mathbb{R}_+ \times K$ such that*

$$\sup_{t>0} \|u(t, \cdot)\|_{L^\infty(K, d\mu)} < \infty.$$

Then u is the Dirichlet Poisson integral of a function $f \in L^\infty(K, d\mu)$.

Proof. Let u be a Dirichlet bounded harmonic function on $\mathbb{R}_+ \times K$, and $M > 0$ such that $|u(t, x)| \leq M$ for every $(t, x) \in \mathbb{R}_+ \times K$.

For $x \in K$ and $n \in \mathbb{Z}_+$, set $f_n(x) = u(1/n, x)$, and let $u_n(t, x)$ be the Dirichlet Poisson integral of f_n . We define then, for $(t, x) \in \mathbb{R}_+ \times K$,

$$U_n(t, x) = u\left(t + \frac{1}{n}, x\right) - u_n(t, x).$$

We claim that $U_n(t, x) \equiv 0$. First note that U_n is bounded, since $|u| \leq M$ and, by Theorem 3.6(1),

$$|u_n(t, x)| \leq AMf_n(x) \leq A\|Mf_n\|_{L^\infty(K)} \leq A\|f_n\|_{L^\infty(K)} \leq AM.$$

Thus $|U_n(t, x)| \leq (A+1)M = A'M$. Moreover, since f_n is continuous, $u_n(t, x)$ can be extended to $t = 0$ with $u_n(0, x) = f_n(x)$, and hence

$$U_n(0, x) = u\left(\frac{1}{n}, x\right) - f_n(x) = 0$$

for every $x \in K$. Fix $(t_0, x_0) \in \mathbb{R}_+ \times K$ and $\varepsilon > 0$ such that $\frac{1}{\varepsilon} > t_0$. Define, for $(t, x) \in [0, \infty) \times K$,

$$U(t, x) = U_n(t, x) + A'M\varepsilon t.$$

Then U is continuous in $[0, \infty) \times K$ and harmonic on $\mathbb{R}_+ \times K$. Moreover, for $t = 0$, $U(0, x) = 0$ for every $x \in K$ and, for $t = \frac{1}{\varepsilon}$,

$$U\left(\frac{1}{\varepsilon}, x\right) = U_n\left(\frac{1}{\varepsilon}, x\right) + A'M \geq 0,$$

since $|U_n(t, x)| \leq A'M$. Finally, as $U_n(t, p) = 0$ for $p \in V_0$, we have

$$U(t, p) = A'Mt\varepsilon \geq 0$$

for $p \in V_0$. Therefore, by Corollary 4.3, $U(t, x) \geq 0$, and hence

$$U_n(t_0, x_0) \geq -A'M\varepsilon t_0.$$

Since ε is arbitrary, $U_n(t_0, x_0) \geq 0$.

Similarly, taking $-U_n(t, x)$, we can conclude that $U_n(t_0, x_0) \leq 0$, and therefore $U_n(t_0, x_0) = 0$ for any $(t_0, x_0) \in [0, \infty) \times K$.

This shows that, for any n , $u(t + 1/n, x)$ is the Dirichlet Poisson integral of f_n , *i. e.*

$$u\left(t + \frac{1}{n}, x\right) = \int_K P^D(t, x, y) f_n(y) d\mu(y).$$

Now, for every n , $\|f_n\|_{L^\infty} \leq M$, so by the weak-* compactness of the ball in $L^\infty(K, d\mu)$, there is a subsequence $f_{n_k} \rightarrow f$ weakly in $L^\infty(K, d\mu)$, and hence

$$\int_K \psi(y) f_{n_k}(y) d\mu(y) \rightarrow \int_K \psi(y) f(y) d\mu(y)$$

for any $\psi \in L^1(K, d\mu)$. Taking, for each $t > 0$ and $x \in K$, $\psi(y) = P^D(t, x, y)$, we obtain, by the continuity of $u(t, x)$, that

$$u(t, x) = \int_K P^D(t, x, y) f(y) d\mu(y).$$

Therefore $u(t, x)$ is the Dirichlet Poisson integral of the L^∞ function f . \square

So we finally have, from Theorems 3.10 and 4.5, the following corollary.

Corollary 4.6. *Then a bounded Dirichlet harmonic function on $\mathbb{R}_+ \times K$ has non-tangential limit at x as $t \rightarrow 0$ for almost every $x \in K$.*

As in the classical setting [Ste70, Chapter VII], Theorem 4.5 can be extended to the case $1 \leq p < \infty$. Together with Theorem 3.6 and Remark 3.9, we obtain the following Corollary.

Corollary 4.7. *Suppose u is a Dirichlet harmonic function on $\mathbb{R}_+ \times K$. If, $1 < p \leq \infty$, then u is the Poisson integral of some $f \in L^p$ if and only if*

$$\sup_{t>0} \|u(t, \cdot)\|_{L^p} < \infty.$$

Moreover, u is the Poisson integral of some finite Borel measure on K if and only if

$$\sup_{t>0} \|u(t, \cdot)\|_{L^1} < \infty.$$

Proof. If u is the Dirichlet Poisson integral of either some $f \in L^p$ or a finite Borel measure on K , the estimates follow from Theorem 3.6 and Remark 3.9.

Now, suppose $p < \infty$ and $\sup_{t>0} \|u(t, \cdot)\|_{L^p} < \infty$. Since $\{\mathcal{P}_t^D\}_{t>0}$ is a semigroup, for any $t > t_0 > 0$,

$$|u(t, x)| \leq \int_K P^D(t - t_0, x, y) |u(t_0, y)| d\mu(y) \lesssim \left(\int_K |P^D(t - t_0, x, y)|^q d\mu(y) \right)^{1/q},$$

where q is the conjugate exponent to p , independently of t_0 because $\|u(t, \cdot)\|_{L^p}$ is uniformly bounded in $t > 0$. From Lemma 3.7, and the same decomposition as in the proof of Theorem 3.6, one proves

$$\left(\int_K |P^D(t - t_0, x, y)|^q d\mu(y) \right)^{1/q} \lesssim (t - t_0)^{-\frac{2d}{d+1} \cdot \frac{1}{p}},$$

uniformly in t , t_0 and x , so we have $u(t, x) \lesssim t^{-\frac{2d}{d+1} \cdot \frac{1}{p}}$ uniformly in x . By Theorem 4.5, $u(t + 1/k, x) = \mathcal{P}^D f_k(x)$, where $f_k(x) = u(1/k, x)$. Since, by assumption, the L^p norms of f_k are uniformly bounded, the Corollary follows from a similar weak-* argument as in the proof of Theorem 4.5. \square

5. A LOCAL FATOU THEOREM

In this section we prove a local Fatou theorem, analogous to the classical nontangential convergence at the boundary of *nontangentially bounded* harmonic functions ([Ste70, Thm. VII.3], [ABR01, Thm. 7.30]).

As we'll need estimates from below for the Neumann Poisson kernel, we can only prove these for the so called *nested* fractals [Lin90], which we define below.

5.1. Nested fractals. We begin by defining the concept of affine nested fractals. We now assume that $(K, S, \{F_i\}_{i \in S})$ is a connected post-critically finite self-similar set, $K \subset \mathbb{R}^n$ and each F_i is the restriction to \mathbb{R}^n of a similitude $F_i : \mathbb{R}^n \rightarrow \mathbb{R}^n$, that is, a map of the form $x \mapsto cUx + a$, where $0 < c < 1$, $a \in \mathbb{R}^n$ and $U \in O(n)$.

We say that a homeomorphism $f : K \rightarrow K$ is a *symmetry* of $(K, S, \{F_i\}_{i \in S})$ if, for every $m \geq 0$, there exists a map $f_m : W_m \rightarrow W_m$ such that

$$f(F_w(V_0)) = F_{f_m(w)}(V_0), \quad \text{for every } w \in W_m.$$

That is, a symmetry preserves the self-similar structure of K .

For $x, y \in \mathbb{R}^n$, let H_{xy} be the bisecting hyperplane of the segment from x to y , and $\psi_{xy} : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be the reflection with respect to H_{xy} .

Definition 5.1.1. We say that $(K, S, \{F_i\}_{i \in S})$ is an *affine nested fractal* if $\psi_{xy}|_K$ is a symmetry for any $x, y \in V_0$.

In other words, reflecting K by each pair of points in its boundary V_0 preserves its self-similar structure. Examples of affine nested fractals are the Sierpinski gasket, the Vicsek set or the pentakun, among others (see [Kig01, Section 3.8] for more examples and a thorough discussion of affine nested fractals).

It can be proven [Kig01, Section 3.8] that if $(K, S, \{F_i\}_{i \in S})$ is an affine nested fractal, then there exist a harmonic structure with all r_i equal to each other, say $r_i = r$, $i \in S$. We assume this harmonic structure is regular, i. e. $r < 1$.

The main result of interest to us is the following theorem [FHK94].

Theorem 5.1. *If H^N is the Neumann heat kernel on the affine nested fractal K , there exist positive constants c_1, c_2, c_3, c_4 such that*

$$(5.1) \quad c_1 t^{-d/(d+1)} \exp \left(-c_2 \left(\frac{R(x, y)^{d+1}}{t} \right)^{1/(d_w-1)} \right) \leq H^N(t, x, y) \\ \leq c_3 t^{-d/(d+1)} \exp \left(-c_4 \left(\frac{R(x, y)^{d+1}}{t} \right)^{1/(d_w-1)} \right),$$

where d_w is the walk dimension with respect to shortest path metric.

A discussion on the shortest path metric and walk dimension can be found in [Bar98]. From Theorem 5.1 we obtain the following integral estimate for the Neumann Poisson kernel.

Corollary 5.2. *Let P^N be the Neumann Poisson kernel on the affine nested fractal K , and $\alpha > 0$. Then there exists $c_\alpha > 0$ such that*

$$\int_B P^N(t, x, y) d\mu(y) \geq c_\alpha,$$

where $B = B_{(\alpha t^2)^{1/(d+1)}}(x)$ is the ball of radius $(\alpha t^2)^{\frac{1}{d+1}}$ with center in x .

Proof. From Theorem 5.1 we have

$$\begin{aligned} P^N(t, x, y) &= \frac{t}{2\sqrt{\pi}} \int_0^\infty e^{-t^2/4s} H^N(s, x, y) \frac{ds}{s^{3/2}} \\ &\gtrsim t \int_0^\infty e^{-t^2/4s} s^{-\frac{3d+1}{2(d+1)}} e^{-c\left(\frac{R(x,y)^{d+1}}{s}\right)^{1/(dw-1)}} \frac{ds}{s}. \end{aligned}$$

Thus

$$\begin{aligned} \int_B P^N(t, x, y) d\mu(y) &\gtrsim \int_B t \int_0^\infty e^{-t^2/4s} s^{-\frac{3d+1}{2(d+1)}} e^{-c\left(\frac{R(x,y)^{d+1}}{s}\right)^{1/(dw-1)}} \frac{ds}{s} d\mu(y) \\ &\geq t \int_B \int_0^\infty e^{-t^2/4s} s^{-\frac{3d+1}{2(d+1)}} e^{-c\left(\frac{\alpha t^2}{s}\right)^{1/(dw-1)}} \frac{ds}{s} d\mu(y), \end{aligned}$$

since $R(x, y)^{d+1} < \alpha t^2$ for $y \in B$.

Now, after the change of variables $s \mapsto \alpha t^2 s$, we finally obtain

$$\begin{aligned} \int_B P^N(t, x, y) d\mu(y) &\gtrsim t \cdot \alpha^{-\frac{3d+1}{2(d+1)}} t^{-\frac{3d+1}{d+1}} \int_B \int_0^\infty e^{-1/4\alpha s} s^{-\frac{3d+1}{2(d+1)}} e^{-c/s^{1/(dw-1)}} \frac{ds}{s} d\mu(y) \\ &= \alpha^{-\frac{3d+1}{2(d+1)}} t^{-\frac{2d}{d+1}} \int_0^\infty e^{-1/4\alpha s} s^{-\frac{3d+1}{2(d+1)}} e^{-c/s^{1/(dw-1)}} \frac{ds}{s} \int_B d\mu(y) \\ &\geq c_\alpha > 0, \end{aligned}$$

because the integral in s converges (and its positive) and

$$\int_B d\mu(y) = \mu(B_{(\alpha t^2)^{1/(d+1)}}(x)) \sim (\alpha t^2)^{\frac{d}{d+1}},$$

by (2.1). \square

Note that, in fact, $c_\alpha = c \alpha^{1/2}$. Corollary 5.2 allows us to construct a harmonic function, with nontangential limit zero, bounded away from zero at the boundary of a union of truncated cones. We define a *truncated cone* on $\mathbb{R}_+ \times K$, for $h, \alpha > 0$ and $x \in K$, as the set

$$\Gamma_\alpha^h(x) = \{(t, y) \in \mathbb{R}_+ \times K : R(x, y)^{d+1} < \alpha t^2, 0 < y < h\}.$$

Lemma 5.3. *Let $E \subset K$ be a measurable set, $\alpha > 0$, and $\Omega = \bigcup_{x \in E} \Gamma_\alpha^1(x)$. Then*

there exists a positive harmonic function v on $\mathbb{R}_+ \times K$ such that

- (1) $v \geq 1$ on $(\partial\Omega) \cap (\mathbb{R}_+ \times K)$; and
- (2) v has nontangential limit 0 at almost every point of E .

Proof. Let P^N be the Neumann Poisson kernel on K and define the function w on $\mathbb{R}_+ \times K$ by

$$w(t, x) = \int_K P^N(t, x, y) \chi_{K \setminus E}(y) d\mu(y) + t,$$

where $\chi_{K \setminus E}$ is the characteristic function of $K \setminus E$.

We see that $w \geq 0$ and, by Theorem 3.10, w has nontangential limit 0 at almost every point of E , so we need to verify that there exists $\delta > 0$ such that $w \geq \delta$ on $(\partial\Omega) \cap (\mathbb{R}_+ \times K)$. Clearly, $w(1, x) \geq 1$ for every $x \in K$.

Now, we observe that $(t, y) \in \Gamma_\alpha(x)$ if and only if $R(x, y) < (\alpha t^2)^{\frac{1}{d+1}}$, that is $x \in B = B_{(\alpha t^2)^{1/(d+1)}}(y)$, the ball of radius $(\alpha t^2)^{\frac{1}{d+1}}$ with center y . Hence, if $(t, y) \in \partial\Omega$, $x \notin B$ for every $x \in E$, and thus $B \subset K \setminus E$.

We thus obtain, by Corollary 5.2,

$$\begin{aligned} \int_K P^N(t, x, y) \chi_{K \setminus E}(y) d\mu(y) &= \int_{K \setminus E} P^N(t, x, y) d\mu(y) \geq \int_B P^N(t, x, y) d\mu(y) \\ &\geq c_\alpha > 0. \end{aligned}$$

Therefore, if we choose $M = \max\{1/c_\alpha, 1\}$, the function $v = Mw$ satisfies the properties required. \square

5.2. Nontangentially bounded functions.

Definition 5.2.1. Let u be a function on $\mathbb{R}_+ \times K$. We say that u is *nontangentially bounded* at $x \in K$ if u is bounded on some $\Gamma_\alpha^h(x)$.

Note that this definition involves only one truncated cone for each x , while the definition of nontangential limit involves all cones over x , regardless of their aperture.

We also observe that, in the case where u is continuous on $\mathbb{R}_+ \times K$, then u is nontangentially bounded at x if and only if it is bounded in some $\Gamma_\alpha^1(x)$.

We now prove the following theorem, analog to the classical local Fatou Theorem [ABR01]. Remember that we assume that K is an affine nested fractal.

Theorem 5.4. *Let u be harmonic on $\mathbb{R}_+ \times K$ and nontangentially bounded at each point in the set $E \subset K$. Then u has a nontangential limit at almost every point of E .*

Proof. Analogously to the classical case, we prove this theorem in a sequence of steps. For each positive integer k , define the set

$$E_k = \{x \in K : |u(t, y)| \leq k \text{ for } (t, y) \in \Gamma_{1/k}^1(x)\}.$$

As u is continuous on $\mathbb{R}_+ \times K$, each E_k is closed and $E = \bigcup_k E_k$.

Step 1. u is bounded on $\Gamma_\alpha^1(x)$ for every $\alpha > 0$ and for almost every $x \in E_k$.

Observe that, if $x \in E_k$ is in the Lebesgue set of χ_{E_k} , then

$$\lim_{r \rightarrow 0} \frac{\mu(B_r(x) \cap E_k)}{\mu(B_r(x))} = 1.$$

Recall also that there exist constants $A_1, A_2 > 0$ such that, for sufficiently small $r > 0$ (say, $r < \bar{r}$),

$$A_1 r^d \leq \mu(B_r(x)) \leq A_2 r^d.$$

Let $x \in E_k$ be in the Lebesgue set of χ_{E_k} . As u is continuous on $\mathbb{R}_+ \times K$, it is sufficient to prove that, for each $\alpha > 0$, there exists $h > 0$ such that $\Gamma_\alpha^h(x) \subset \bigcup_{z \in E_k} \Gamma_{1/k}^1(z)$. It is of course sufficient to consider the case $\alpha > 1/k$.

Let $\delta > 0$ such that $\delta < \min\{\bar{r}, k^{-1/(d+1)}\}$ and, for $0 < r \leq \delta$,

$$(5.2) \quad \frac{\mu(B_r(x) \cap E_k)}{\mu(B_r(x))} > 1 - \frac{A_1}{A_2} \left(\frac{k^{-1/(d+1)}}{\alpha^{1/(d+1)} + k^{-1/(d+1)}} \right)^d.$$

Set $h = \left(\frac{\delta}{2\alpha^{1/(d+1)}} \right)^{(d+1)/2}$. Thus $h < 1$ and we shall prove $\Gamma_\alpha^h(x) \subset \Omega$.

We first observe that, if $(t, y) \in \Gamma_\alpha^h(x)$, then

$$B_{(t^2/k)^{1/(d+1)}}(y) \cap E \neq \emptyset.$$

Indeed, if $(t, y) \in \Gamma_\alpha^h(x)$, then $R(x, y) < (\alpha t^2)^{1/(d+1)}$ and $0 < t < h$. Now, for $z \in B_{(t^2/k)^{1/(d+1)}}(y)$, $R(y, z) < (t^2/k)^{1/(d+1)}$ and hence

$$R(x, z) < (\alpha^{1/(d+1)} + k^{-1/(d+1)})t^{2/(d+1)},$$

so $z \in B_r(y)$, where $r = (\alpha^{1/(d+1)} + k^{-1/(d+1)})t^{2/(d+1)}$. Therefore

$$B_{(t^2/k)^{1/(d+1)}}(y) \subset B_r(y).$$

If $B_{(t^2/k)^{1/(d+1)}}(y) \cap E_k = \emptyset$, then $B_r(x) \cap E_k \subset B_r(x) \setminus B_{(t^2/k)^{1/(d+1)}}(y)$ and thus

$$\begin{aligned} \frac{\mu(B_r(x) \cap E_k)}{\mu(B_r(x))} &\leq \frac{\mu(B_r(x) \setminus B_{(t^2/k)^{1/(d+1)}}(y))}{\mu(B_r(x))} = 1 - \frac{\mu(B_{(t^2/k)^{1/(d+1)}}(y))}{\mu(B_r(x))} \\ &\leq 1 - \frac{A_1((t^2/k)^{1/(d+1)})^d}{A_2 r^d} = 1 - \frac{A_1}{A_2} \left(\frac{k^{-1/(d+1)}}{\alpha^{1/(d+1)} + k^{-1/(d+1)}} \right)^d, \end{aligned}$$

which contradicts (5.2) since

$$r = (\alpha^{1/(d+1)} + k^{-1/(d+1)})t^{2/(d+1)} < (\alpha^{1/(d+1)} + k^{-1/(d+1)})h^{2/(d+1)} < \delta,$$

by the choice of h .

Hence there exists $x_0 \in B_{(t^2/k)^{1/(d+1)}}(y) \cap E_k$, which implies $(t, y) \in \Gamma_{1/k}^1(x_0)$, as desired. This finishes the proof of Step 1.

Therefore, there is a subset $F \subset E$ such that $\mu(E \setminus F) = 0$ and u is bounded in every cone $\Gamma_\alpha^1(x)$ for $\alpha > 0$ and $x \in F$.

In particular, for a fixed $\alpha > 0$, u is bounded in every $\Gamma_\alpha^1(x)$ for every $x \in F$. We can thus write $F = \bigcup_k F_k$, with

$$F_k = \{x \in F : |u(t, y)| \leq k \text{ for } (t, y) \in \Gamma_\alpha^1(x)\}.$$

Step 2. At almost every $x \in F_k$, the limit

$$\lim_{\substack{(t, y) \rightarrow (0, x) \\ (t, y) \in \Gamma_\alpha(x)}} u(t, y)$$

exists. That is, u has a limit at the boundary point x within the cone $\Gamma_\alpha(x)$.

Let $\Omega = \bigcup_{x \in F_k} \Gamma_\alpha^1(x)$. As u is continuous on $\mathbb{R}_+ \times K$, we may assume, say, $|u| \leq 1$ on the set $\Omega' = \bigcup_{x \in F_k} \Gamma_\alpha^2(x)$. Without loss of generality, we can also assume u is real valued.

Now, for each $n \geq 1$, let

$$G_n = \{y \in K : \text{there exists } x \in F_k \text{ such that } R(x, y)^{d+1} < \alpha/n^2\}.$$

Each G_n is open and $G_n \subset F_k$. Now define the functions f_n on K by

$$f_n(x) = \chi_{G_n}(x)u(1/n, x).$$

As $(1/n, x) \in \Omega$ if and only if $x \in G_n$, we have $|f_n| \leq 1$, and thus, passing to a subsequence, (f_n) weakly-* converges to some $f \in L^\infty(K, d\mu)$, in particular

$$\mathcal{P}_t^N f_n(x) \rightarrow \mathcal{P}_t^N f(x)$$

for each $(t, x) \in \mathbb{R}_+ \times K$, where $\mathcal{P}_t^N f_n$ and $\mathcal{P}_t^N f$ are the Neumann Poisson integrals of the functions f_n and f , respectively. Moreover, the function

$$u_n(t, x) = \mathcal{P}_t^N f_n(x) - u(t + 1/n, x)$$

is harmonic and extends continuously to $\{0\} \times G_n$, because f_n is continuous, with $u_n(0, x) = 0$ for each $x \in G_n$. Moreover, $|u_n| \leq 2$ on the closure of Ω , since $t + 1/n \leq 2$.

If we choose v as in Lemma 5.3, then

$$\liminf_{(t,y) \rightarrow \partial\Omega} (2v \pm u_n)(t, x) \geq 0,$$

so, by the maximum principle (Corollary 4.4), $2v \pm u_n \geq 0$ on Ω . Taking $n \rightarrow \infty$, we conclude that

$$|\mathcal{P}_t^N f(x) \pm u| \leq 2v$$

on Ω and, as $\mathcal{P}_t^N f(x)$ and v have nontangential limits (Theorem 3.10 and Lemma 5.3), we obtain Step 2.

Hence, for each $\alpha > 0$, u has a limit at the boundary, within the cone $\Gamma_\alpha(x)$, for every point x in a subset $F'_\alpha \subset F$ with $\mu(F \setminus F'_\alpha) = 0$.

If we take $F' = \bigcup_k F'_k$, we conclude that u has nontangential limit at every point in F' , with $\mu(E \setminus F') = 0$, as desired. \square

ACKNOWLEDGEMENTS

The author would like to thank the referee for providing useful suggestions, which led to the improvement of some of the results of this paper.

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